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On the canonical algebraic structure of a category

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Abstract

For any locally small category $\mathscr{A}$, applying Lawvere's "structure" functor to the hom-functor $H = \mathrm{Hom}_{\mathscr{A}} : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathbf{Set}$ produces a Lawvere theory $\mathscr{A}^*$, called *the canonical algebraic structure of* $\mathscr{A}$, and given by $\mathscr{A}^*(n,m) = [\mathscr{A}^{\mathrm{op}} \times \mathscr{A}, \mathbf{Set}](H^n, H^m)$ — provided that this latter set is small, which is certainly the case when $\mathscr{A}$ is complete and cocomplete and some small subset of its objects is either generating or co-generating. If now $\mathscr{T}$ is a *commutative* theory, so that $\mathscr{T}$-Alg is a symmetric monoidal closed category, enrichments of $\mathscr{A}$ over $\mathscr{T}$-Alg correspond to liftings of H through the forgetful functor $U : \mathscr{T}\text{-Alg} \to \mathbf{Set}$, and hence, (by Lawvere's structure-semantics adjunction) to theory-maps $\mathscr{T} \to \mathscr{A}^*$. In fact, whenever $\mathscr{A}$ admits either finite powers or finite multiples, the theory $\mathscr{A}^*$ is itself commutative, so that $\mathscr{A}$ has a canonical enrichment over $\mathscr{A}^*$. When $\mathscr{A}$ is of the form $\mathscr{T}$-Alg for some theory $\mathscr{T}$ we find that $\mathscr{T}\text{-Alg}^* \cong \mathscr{T}^*$, each being isomorphic to the *centre* of $\mathscr{T}$. We end by considering the situation where $\mathscr{A}$ is already enriched over some symmetric monoidal category $\mathscr{V}$, and may in particular be $\mathscr{V}$ itself.

MSC: 18C10; 18C15; 18D20

1. Introduction

As we said in the abstract above, our concern is with the Lawvere theory $\mathscr{A}^*$ given by

$$\mathscr{A}^*(n,m) = [\mathscr{A}^{\mathrm{op}} \times \mathscr{A}, \mathbf{Set}](H^n, H^m)$$

where $H = \mathrm{Hom}_{\mathscr{A}} : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathbf{Set}$; this theory, which in many cases is itself commutative, has the property that the theory-maps $\mathscr{T} \to \mathscr{A}^*$ from a commutative theory $\mathscr{T}$

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are in bijection with enrichments of $\mathscr{A}$ over the symmetric monoidal closed category $\mathscr{T}$-Alg. In fact, $\mathscr{A}$ is properly called a Lawvere theory only when each $\mathscr{A}^*(n,m)$ is a small set; but this is indeed the case if $\mathscr{A}$ or $\mathscr{A}^{\mathrm{op}}$ admits finite powers and a small cogenerating set. Before proceeding we briefly recall the basic facts about Lawvere theories and the structure-semantics adjunction; see Lawvere [5] for the original treatment and Freyd [3] or Borceux [2] for additional expositions of the subject.

Write $\mathbf{S}$ for the full subcategory of $\mathbf{Set}$ whose objects are the natural numbers; since each $n \in \mathbf{S}$ is the multiple $n \cdot 1 = 1 + \cdots + 1$, so in $\mathbf{S}^{\mathrm{op}}$ each n is the power 1^n. By a *Lawvere theory* (henceforth just a *theory*) we mean a small category $\mathscr{T}$ whose objects form the set $\mathbf{N}$ of natural numbers, together with a functor $j_{\mathscr{T}} : \mathbf{S}^{\mathrm{op}} \to \mathscr{T}$ which is the identity on objects and preserves the power 1^n for each n. Equivalently, a theory is a small category with finite powers $\mathscr{T}$ having $\mathrm{ob}(\mathscr{T}) = \mathbf{N}$, together with designated maps $p_i^n : n \to 1$ for $1 \leq i \leq n$ which express n as the power 1^n in $\mathscr{T}$ and satisfy $p_1^1 = 1$ (note that necessarily $n = 1^n$ — in $\mathscr{T}$ — even for $n = 0$). For such a theory we have of course $\mathscr{T}(n,m) \cong \mathscr{T}(n,1)^m$. A morphism (or *map*) $\rho : \mathscr{T} \to \mathscr{S}$ of theories is a functor satisfying $\rho j_{\mathscr{T}} = j_{\mathscr{S}}$; equivalently, a functor $\rho : \mathscr{T} \to \mathscr{S}$ which is the identity on objects and *strictly* preserves the *designated* powers 1^n. We write $\mathbf{Th}$ for the category of theories and their maps; note that $\mathbf{S}^{\mathrm{op}}$ is itself an initial object in this category.

If the locally small category $\mathscr{A}$ admits (chosen) finite powers, each object A of $\mathscr{A}$ determines a theory $\langle A \rangle$ given by $\langle A \rangle(n,m) = \mathscr{A}(A^n, A^m)$. In particular, when $\mathscr{B}$ admits (chosen) finite powers, a functor $F : \mathscr{A} \to \mathscr{B}$, being an object of the functor category $[\mathscr{A}, \mathscr{B}]$, determines a theory $\langle F \rangle$, where

$$\langle F \rangle(n,m) = [\mathscr{A}, \mathscr{B}](F^n, F^m) = \int_{A \in \mathscr{A}} \mathscr{B}((FA)^n, (FA)^m) \tag{1}$$

is the set of natural transformations $F^n \to F^m$; provided that this is a small set for each n and m, in which case the functor F is said to be *tractable*. Right adjoint functors are always tractable: we return to this below in the particular case $\mathscr{B} = \mathbf{Set}$.

For an object A of a category with (chosen) finite multiples, we shall later on find it convenient to write $\{A\}$ for the theory which would be written as $\langle A \rangle$ when A is seen as an object of $\mathscr{A}^{\mathrm{op}}$; so that $\{A\}(n,m) = \mathscr{A}(m \cdot A, n \cdot A)$.

If $\mathscr{A}$ is a category with (chosen) finite powers, a *model in $\mathscr{A}$ of the theory $\mathscr{T}$* is a functor $M : \mathscr{T} \to \mathscr{A}$ which strictly preserves the chosen finite powers; such models, with natural transformations as morphisms, form a full subcategory $\mathrm{Mod}[\mathscr{T}, \mathscr{A}]$ of the functor category $[\mathscr{T}, \mathscr{A}]$, with a faithful and conservative forgetful functor sending M to $M1$. So in particular a theory-map $\rho : \mathscr{T} \to \mathscr{S}$ is a model of $\mathscr{T}$ in $\mathscr{S}$; on the other hand, to give a model $M : \mathscr{T} \to \mathscr{A}$ is equally to give the object $A = M1$ of $\mathscr{A}$ and a theory map $\mathscr{T} \to \langle A \rangle$.

Turning to the special case $\mathscr{A} = \mathbf{Set}$ we write $\mathscr{T}$-Alg for $\mathrm{Mod}[\mathscr{T}, \mathbf{Set}]$; a $\mathscr{T}$-model in $\mathbf{Set}$ is callled a *$\mathscr{T}$-algebra*, and a morphism of $\mathscr{T}$-algebras is called a *homomorphism of algebras*. We denote the forgetful functor $\mathscr{T}\text{-Alg} \to \mathbf{Set}$ by $U^{\mathscr{T}}$, or by U for short. Composition with a theory-map $\rho : \mathscr{T} \to \mathscr{S}$ induces a functor $\rho^* : \mathscr{S}\text{-Alg} \to \mathscr{T}\text{-Alg}$ with

$U^{\mathscr{T}}\rho^* = U^{\mathscr{S}}$. In this way we obtain a *semantics* functor $\mathbf{Sem} : \mathbf{Th}^{\mathrm{op}} \to \mathbf{CAT}/\mathbf{Set}$ sending $\mathscr{T}$ to $U^{\mathscr{T}} : \mathscr{T}\text{-Alg} \to \mathbf{Set}$; here $\mathbf{CAT}$ is the category of locally small categories, and a morphism in $\mathbf{CAT}/\mathbf{Set}$ from $V : \mathscr{A} \to \mathbf{Set}$ to $V' : \mathscr{A}' \to \mathbf{Set}$ is a functor $P : \mathscr{A} \to \mathscr{A}'$ giving an *equality* $V'P = V$.

As in the penultimate paragraph, to give a $\mathscr{T}$-algebra A is to give the set A and the theory map $\mathscr{T} \to \langle A \rangle$. The functions $\mathscr{T}(n,m) \to \mathbf{Set}(A^n, A^m)$ which constitute this last are of course fully determined by the $\mathscr{T}(n,1) \to \mathbf{Set}(A^n, A)$; and to give these is to give a sequence of functions $a_n : \mathscr{T}(n,1) \times A^n \to A$ which is natural in n and satisfies the conditions asserting the functoriality of $\mathscr{T} \to \mathbf{Set}$. One easily sees that the endofunctor $T : \mathbf{Set} \to \mathbf{Set}$ given by

$$TA = \int^{n \in \mathbf{S}} \mathscr{T}(n,1) \times A^n \tag{2}$$

has a canonical monad-structure induced by the composition and the identities of $\mathscr{T}$, and that the function $a : TA \to A$ given by the (a_n) above is an *action* of T on A precisely when A is a $\mathscr{T}$-algebra. Thus, we have an isomorphism $\mathscr{T}\text{-Alg} \cong T\text{-Alg}$, where T-Alg is the category of algebras for the monad T; and this isomorphism identifies $U^{\mathscr{T}}$ with the usual forgetful functor $U^T : T\text{-Alg} \to \mathbf{Set}$. We conclude that $U^{\mathscr{T}}$ has a left adjoint $F^{\mathscr{T}} : \mathbf{Set} \to \mathscr{T}\text{-Alg}$, where $F^{\mathscr{T}}A$ is TA with the appropriate action. In particular, since the Yoneda isomorphism applied to (2) gives

$$Tn \cong \mathscr{T}(n,1) \tag{3}$$

for $n \in \mathbf{S}$, we may see $\mathscr{T}(n,1)$ as *the free* $\mathscr{T}$*-algebra on* n. (Note here that we have implicitly supposed $\mathscr{T}$ to be a small category; were it not so, the Tn of (3) would be a large set, and could not serve as the value of a left adjoint $F^{\mathscr{T}}$ to $U^{\mathscr{T}}$. Since the functor $[\mathscr{T}, \mathbf{Set}] \to \mathbf{Set}$ given by evaluation at 1 *does* have a left adjoint, the lack of a left adjoint to $U^{\mathscr{T}}$ implies the lack of a left adjoint to the inclusion $\mathscr{T}\text{-Alg} \to [\mathscr{T}, \mathbf{Set}]$; so that in the case of a large $\mathscr{T}$ we lose the usual proof that $\mathscr{T}$-Alg is cocomplete. It is important, therefore, to pay attention to the smallness requirement).

It is harmless to treat (3) as an equality. We may call the elements ω of $Tn = \mathscr{T}(n,1)$ the *n-ary operations* of the theory; indeed, for an algebra A, the function

$$a_n : \mathscr{T}(n,1) \times A^n \to A$$

above sends $(\omega, x_1, \dots, x_n)$, where $\omega \in \mathscr{T}(n,1)$, to the *value* of the operation, often written as $\omega_A(x_1, \dots, x_n)$. Of course, we see an element of $\mathscr{T}(n,m) = \mathscr{T}(n,1)^m$ as an *m-ad of n-ary operations*.

Consider the extent to which the functor $\mathbf{Sem} : \mathbf{Th}^{\mathrm{op}} \to \mathbf{CAT}/\mathbf{Set}$ has a left adjoint. To give a morphism in $\mathbf{CAT}/\mathbf{Set}$ from $V : \mathscr{A} \to \mathbf{Set}$ to $U^{\mathscr{T}} : \mathscr{T}\text{-Alg} \to \mathbf{Set}$ is just to give to each VA a $\mathscr{T}$-algebra structure in such a way that each Vf is a homomorphism. This is equivalent to giving a $\mathscr{T}$-model structure to the object V of $[\mathscr{A}, \mathbf{Set}]$, and hence to giving a theory-map $\mathscr{T} \to \langle V \rangle$. Of course $\langle V \rangle$ is, as we said, an honest theory only when V is tractable, in the sense that each $[\mathscr{A}, \mathbf{Set}](V^n, V^m)$ is small; for which it suffices that each $[\mathscr{A}, \mathbf{Set}](V^n, V)$ be small, since $[\mathscr{A}, \mathbf{Set}](V^n, V^m) =$

$([\mathscr{A},\mathbf{Set}](V^n,V))^m$. Observe that V is certainly tractable if it has a left adjoint G; for then $V \cong \mathscr{A}(I,-)$ where $I = G1$, so that the Yoneda isomorphism gives

$$\begin{aligned}[\mathscr{A},\mathbf{Set}](V^n,V) &\cong [\mathscr{A},\mathbf{Set}](\mathscr{A}(n\cdot I,-),\mathscr{A}(I,-))\\ &\cong \mathscr{A}(I,n\cdot I)\\ &\cong VGn. \end{aligned} \tag{4}$$

It follows that the functor $\mathbf{Sem}:\mathbf{Th}^{\mathrm{op}} \to \mathbf{CAT}/\mathbf{Set}$ takes its values in the full subcategory $\mathbf{TRACT}/\mathbf{Set}$ of $\mathbf{CAT}/\mathbf{Set}$ determined by the tractable functors (and even in the still smaller full subcategory $\mathbf{RADJ}/\mathbf{Set}$ determined by the right-adjoint functors); and that $\mathbf{Sem}$, seen now as a functor $\mathbf{Th}^{\mathrm{op}} \to \mathbf{TRACT}/\mathbf{Set}$, has a left adjoint $\mathbf{Str}:\mathbf{TRACT}/\mathbf{Set} \to \mathbf{Th}^{\mathrm{op}}$ sending $V:\mathscr{A}\to\mathbf{Set}$ to the theory $\langle V\rangle$, and known (following its discoverer Lawvere) as the *structure* functor. In fact $\mathbf{Sem}$ is fully faithful, since the counit of the adjunction $\mathbf{Sem} \dashv \mathbf{Str}$ is invertible: by (4), the forgetful $U^{\mathscr{T}}:\mathscr{T}\text{-Alg}\to\mathbf{Set}$ has $\langle U^{\mathscr{T}}\rangle(n,1) \cong U^{\mathscr{T}}F^{\mathscr{T}}n = Tn$, so that $\langle U^{\mathscr{T}}\rangle \cong \mathscr{T}$ by (3). In fact, the monadic functor $U^{\mathscr{T}}$ is *finitary*, in the sense that it preserves filtered colimits; and it is well known that the image of the fully faithful semantics functor consists precisely of the finitary right-adjoint functors into $\mathbf{Set}$. Accordingly $\mathbf{Str}$ may be seen as a reflection of $\mathbf{TRACT}/\mathbf{Set}$ into the full subcategory determined by such functors.

Following Linton [6] we call a theory $\mathscr{T}$ *commutative* when, for each algebra A and each operation $\omega\in\mathscr{T}(n,1)$, the function $\omega_A:A^n\to A$ is a homomorphism of $\mathscr{T}$-algebras. It is equivalent to require, for each $\omega\in\mathscr{T}(n,1)$ and each $\theta\in\mathscr{T}(m,1)$, the commutativity in $\mathscr{T}$ of the diagram

$$\begin{tikzcd}
(1^n)^m = n^m \arrow[r, "\omega^m"] \arrow[d, "c"', "\cong"] & 1^m = m \arrow[dr, "\theta"] & \\
 & & 1 \\
(1^m)^n = m^n \arrow[r, "\theta^n"'] & 1^n = n \arrow[ur, "\omega"'] &
\end{tikzcd} \tag{5}$$

where c is the evident isomorphism. More generally, when (5) commutes for the particular operations ω and θ, we say that ω and θ *commute*. An operation ω that commutes with itself is sometimes said to be *autonomous*. An operation ω is said to be *central* when it commutes with *every* operation of $\mathscr{T}$; it is easy to see (and in any case will become clear below) that these central operations are themselves the operations of a new theory $Z(\mathscr{T})$ (a subtheory of $\mathscr{T}$) called the *centre* of $\mathscr{T}$. (Of course the centre $Z(\mathscr{T})$ of the theory $\mathscr{T}$, which is itself a theory, is not to be confused with *the centre of $\mathscr{T}$ as a category*, which is a set, or rather a commutative monoid: recall that the centre $\mathscr{Z}(\mathscr{A})$ of a category $\mathscr{A}$ is the monoid of endomorphisms of the identity functor $1_{\mathscr{A}}:\mathscr{A}\to\mathscr{A}$, given by $\int_{A\in\mathscr{A}}\mathscr{A}(A,A)$.)

As Linton [6] first observed, $\mathscr{T}$-Alg for a commutative $\mathscr{T}$ has a canonical symmetric monoidal closed structure: here the internal-hom $[B,C]$ is the set $\mathscr{T}\text{-Alg}(B,C)$ of homomorphisms, made into a $\mathscr{T}$-algebra as a subalgebra of the power-algebra C^{UB}, while the tensor product $A\otimes B$ represents the *bi-homomorphisms* — that is, the functions $A\times B\to C$ which are homomorphisms in each variable separately when the other is fixed; and the unit I for the tensor product is the free algebra $F1$ on one element, which represents the forgetful functor $U:\mathscr{T}\text{-Alg}\to\mathbf{Set}$.

Finally we remark that, instead of considering as our structure functor the left adjoint of the semantics functor $\mathbf{Th}^{\mathrm{op}}\to\mathbf{CAT}/\mathbf{Set}$, we could equally have considered the left adjoint of its extension $\mathbf{Sem}:\mathbf{Mnd}^{\mathrm{op}}\to\mathbf{CAT}/\mathbf{Set}$, where $\mathbf{Mnd}$ is the category of all monads on $\mathbf{Set}$ (while $\mathbf{Th}$ is in effect the category of finitary ones). This extended $\mathbf{Sem}$ sends the monad T to $U^T:T\text{-Alg}\to\mathbf{Set}$, and its left adjoint sends $V:\mathscr{A}\to\mathbf{Set}$ to the monad $\langle\langle V\rangle\rangle=\mathrm{Ran}_V V$ given by right Kan extension of V along itself, whenever this exists; then the theory $\langle V\rangle$ is just the further reflexion of $\langle\langle V\rangle\rangle$ into the *finitary* monads. Note that $\mathrm{Ran}_V V$ is just the monad VG when V has a left adjoint G. It is because commutative theories are probably better known than commutative monads that we have elected to emphasize $\langle V\rangle$ rather than $\langle\langle V\rangle\rangle$, but the results below remain essentially the same in either formulation.

This completes our revision of the classical results, and we turn now to our observations.

2. An observation concerning enrichments

A $\mathscr{V}$-category $\mathbf{A}$, where $\mathscr{V}$ is a symmetric monoidal category, is said to be an *enrichment* of the ordinary category $\mathscr{A}$ when the underlying ordinary category $\mathbf{A}_0$ of $\mathbf{A}$ is $\mathscr{A}$. The matter of enrichment over $\mathscr{V}$ becomes particularly simple when $\mathscr{V}$ is of the form $\mathscr{T}$-Alg for some commutative theory $\mathscr{T}$; for then to enrich $\mathscr{A}$ over $\mathscr{V}$ we need only produce a functor $\mathbf{A}(-,-):\mathscr{A}^{\mathrm{op}}\times\mathscr{A}\to\mathscr{T}\text{-Alg}$ which *lifts* $H=\mathrm{Hom}_{\mathscr{A}}:\mathscr{A}^{\mathrm{op}}\times\mathscr{A}\to\mathbf{Set}$ in the sense that we have (strict) commutativity in

$$\begin{tikzcd} \mathscr{A}^{\mathrm{op}}\times\mathscr{A} \arrow[r, "\mathrm{A}(-,-)"] \arrow[dr, "H"'] & \mathscr{T}\text{-Alg} \arrow[d, "U"] \\ & \mathbf{Set}. \end{tikzcd} \tag{6}$$

That (6) is necessary is well known; see for example [4, (1.33)]. In the present case it is also sufficient: for there are unique maps $M:\mathbf{A}(B,C)\otimes\mathbf{A}(A,B)\to\mathbf{A}(A,C)$ and $j:I\to\mathbf{A}(A,A)$ making $\mathbf{A}$ into a ($\mathscr{T}$-Alg)-category with underlying ordinary category $\mathscr{A}$. Here j is the image of 1_A under the isomorphism $\mathscr{A}(A,A)=U\mathbf{A}(A,A)\cong\mathscr{T}\text{-Alg}(I,\mathbf{A}(A,A))$, while M corresponds to the composition map $\mathscr{A}(B,C)\times\mathscr{A}(A,B)\to\mathscr{A}(A,C)$; this map is indeed a homomorphism in each variable when the other is fixed, since $\mathscr{A}(A,g)$ underlies $\mathbf{A}(A,g):\mathbf{A}(A,B)\to\mathbf{A}(A,C)$ and $\mathscr{A}(f,C)$ underlies $\mathbf{A}(f,C):\mathbf{A}(B,C)\to\mathbf{A}(A,C)$.

To give a functor $\mathbf{A}(-,-)$ as in (6) is just to give a morphism $H\to U^{\mathscr{T}}=\mathbf{Sem}\,\mathscr{T}$ in $\mathbf{CAT}/\mathbf{Set}$. By the structure-semantics adjunction this is equally to give a morphism $\mathscr{T}\to\mathbf{Str}\,H=\langle H\rangle$ in the category $\mathbf{Th}$ of theories. Henceforth we write $\mathscr{A}^*$ for the theory $\langle H\rangle$ and call it the *canonical algebraic structure* of $\mathscr{A}$; of course it is an honest theory only when H is tractable. Stated formally:

Proposition 1. *For a commutative theory $\mathscr{T}$, there is a bijection between enrichments over $\mathscr{T}$-*Alg *of a category $\mathscr{A}$ and theory-maps $\mathscr{T}\to\mathscr{A}^*$ into the canonical algebraic structure $\mathscr{A}^*=\langle H\rangle$ of $\mathscr{A}$.*

Remark 2. (a) Recall that a *Maltsev operation* on a set A is a map $m:A^3\to A$ satisfying $m(x,y,y)=x$ and $m(x,x,y)=y$; it is a *natural Maltsev operation* if it is fact autonomous, which is to say that it satisfies the equation

$$\begin{aligned}&m(m(x_1,y_1,z_1),m(x_2,y_2,z_2),m(x_3,y_3,z_3))\\&\quad=m(m(x_1,x_2,x_3),m(y_1,y_2,y_3),m(z_1,z_2,z_3)).\end{aligned}$$

A set A with a natural Maltsev operation is variously called an *abelian Maltsev algebra* or an *affine space over the integers*; let us write **Aff** for the variety of such algebras. It follows from Proposition 1 that the enrichments (if any) of a category $\mathscr{A}$ over **Aff** correspond to the natural Maltsev operations in $\mathscr{A}^*$.

(b) The variety freely generated by a natural Maltsev operation and a constant is the variety **Ab** of a abelian groups. So a natural Maltsev operation in $\mathscr{A}^*$ together with a constant in $\mathscr{A}^*$ produce an enrichment of $\mathscr{A}$ over **Ab** — that is, an additive (some say “pre-additive”) structure on $\mathscr{A}$.

(c) Any category $\mathscr{A}$ has a canonical enrichment over $\mathscr{T}$-Alg, where $\mathscr{T}$ is any commutative sub-theory of $\mathscr{A}^*$, such as the centre $Z(\mathscr{A}^*)$ of $\mathscr{A}^*$.

(d) We shall see in Proposition 9 that the theory $\mathscr{A}^*$ is itself commutative if $\mathscr{A}$ admits finite powers or finite multiples. It follows that such an $\mathscr{A}$ has a canonical enrichment over $\mathscr{A}^*$-Alg. For a general $\mathscr{A}$, this need not tell us much — for it may well be the case that $\mathscr{A}^*=\mathbf{S}^{\mathrm{op}}$, and then $\mathscr{A}^*$-Alg $=$ **Set**. The cases of interest, of course, are those where $\mathscr{A}^*$ does not reduce to $\mathbf{S}^{\mathrm{op}}$.

(e) By a ternary version of the Eckmann–Hilton argument, two Maltsev operations m, m' on a set coincide if each commutes with the other. Accordingly there is at most one enrichment of $\mathscr{A}$ over **Aff** when the theory $\mathscr{A}^*$ is commutative, and a fortiori at most one enrichment of $\mathscr{A}$ over **Ab**. This last is classical when $\mathscr{A}$ has finite products or coproducts; but by the above it still holds when $\mathscr{A}$ only admits finite powers or finite copowers. Recall that it is not true for all $\mathscr{A}$: of the four-element rings, two have isomorphic multiplicative structure, but non-isomorphic additive structure.

3. General observations on $\mathscr{A}^*$

Since $\mathscr{A}^*(n,m)=(\mathscr{A}^*(n,1))^m$, it suffices to consider the set $\mathscr{A}^*(n,1)$ of n-ary operations of $\mathscr{A}^*$, given as in (1) by the set

$$\mathscr{A}^*(n,1)=\int_{A,B\in\mathscr{A}}\mathbf{Set}(\mathscr{A}(A,B)^n,\mathscr{A}(A,B)) \tag{7}$$

of natural transformations

$$\mu_{AB} : \mathscr{A}(A, B)^n \to \mathscr{A}(A, B). \tag{8}$$

There is an alternative way of writing the above, in terms of the Yoneda embedding $Y : \mathscr{A} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ given by $YB = \mathscr{A}(-, B)$: since limits — and in particular powers — in $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ are formed pointwise, as are those in $[\mathscr{A}, [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]]$, we have

$$\begin{aligned}
\mathscr{A}^*(n,1) &= \int_{A,B} \mathbf{Set}(\mathscr{A}(A, B)^n, \mathscr{A}(A, B)) \\
&= \int_{A,B} \mathbf{Set}(((YB)A)^n, (YB)A) \\
&= \int_{A,B} \mathbf{Set}(((YB)^n)A, (YB)A) \\
&= \int_{B} [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]((YB)^n, YB) \\
&= [\mathscr{A}, [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]](Y^n, Y) \\
&= \langle Y \rangle(n,1);
\end{aligned}$$

thus in fact we have

$$\mathscr{A}^* = \langle Y \rangle \tag{9}$$

and $\mathscr{A}^*(n,1)$ is the set of natural transformations $Y^n \to Y$.

Since (8) is unchanged when we interchange A and B, we have:

Proposition 3. *There is a canonical isomorphism* $(\mathscr{A}^{\mathrm{op}})^* \cong \mathscr{A}^*$.

In general, a functor $J : \mathscr{A} \to \mathscr{B}$ induces neither a theory map $\mathscr{A}^* \to \mathscr{B}^*$ nor a theory map $\mathscr{B}^* \to \mathscr{A}^*$; the passage from $\mathscr{A}$ to $\mathscr{A}^*$ is not functorial. When J is *fully faithful*, however, there is a theory map $J^* : \mathscr{B}^* \to \mathscr{A}^*$ given by *restriction*: a natural transformation $v_{CD} : \mathscr{B}(C,D)^n \to \mathscr{B}(C,D)$ restricts to a natural transformation $v_{JA,JB} : \mathscr{B}(JA,JB)^n \to \mathscr{B}(JA,JB)$, and $\mathscr{B}(JA,JB) \cong \mathscr{A}(A, B)$. In one important case this J^* is invertible. Recall that the *Cauchy completion* of $\mathscr{A}$ (also called the *Karoubi envelope* of $\mathscr{A}$), which is the closure of $\mathscr{A}$ in $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ under the splitting of idempotents, is the following category $\mathscr{B}$: an object of $\mathscr{B}$ is a pair (A,e) where $A \in \mathscr{A}$ and e is an idempotent endomorphism of A, while a morphism $(A,e) \xrightarrow{u} (B,f)$ in $\mathscr{B}$ is a morphism $A \xrightarrow{u} B$ in $\mathscr{A}$ satisfying $fue = u$.

Proposition 4. *When* $J : \mathscr{A} \to \mathscr{B}$ *is the inclusion of* $\mathscr{A}$ *into its Cauchy completion, the theory-map* $J^* : \mathscr{B}^* \to \mathscr{A}^*$ *is invertible.*

Proof. If $\mu \in \mathscr{B}(n,1)$ has components $\mu_{(A,e),(B,f)} : \mathscr{B}((A,e),(B,f))^n \to \mathscr{B}((A,e),(B,f))$, then $J^*\mu = \mu^-$ where $(\mu^-)_{AB} : \mathscr{A}(A, B)^n \to \mathscr{A}(A, B)$ is $\mu_{(A,1_A),(B,1_B)}$. From any $\tau \in \mathscr{A}^*(n,1)$, with components $\tau_{AB} : \mathscr{A}(A, B)^n \to \mathscr{A}(A, B)$, we obtain an element τ^+ of $\mathscr{B}^*(n,1)$ by taking $(\tau^+)_{(A,e),(B,f)}$ to be τ_{AB}; this is possible since τ_{AB} maps the subset $\mathscr{B}((A,e),(B,f))^n$ of $\mathscr{A}(A, B)^n$ into the subset $\mathscr{B}((A,e),(B,f))$ of $\mathscr{A}(A, B)$ — the point being that $fu_ie=u_i$ gives $f\tau(u_1,\dots,u_n)e=\tau(fu_1e,\dots,fu_ne)=\tau(u_1,\dots,u_n)$ — and since τ^+ like τ is natural. Clearly $\tau^{+-}=\tau$, so that it remains only to prove that $\mu^{-+}=\mu$, which is the assertion that, for $u_i\in\mathscr{B}((A,e),(B,f))$, we have

$$\mu_{(A,1_A),(B,1_B)}(u_1,\dots,u_n)=\mu_{(A,e),(B,f)}(u_1,\dots,u_n). \tag{10}$$

However the naturality of μ applied to the maps $(A,1)\xrightarrow{e}(A,e)$ and $(B,f)\xrightarrow{f}(B,1)$ of $\mathscr{B}$ gives

$$f\mu_{(A,e),(B,f)}(u_1,\dots,u_n)e=\mu_{(A,1_A),(B,1_B)}(fu_1e,\dots,fu_ne), \tag{11}$$

while the naturality of μ applied to the maps $(A,e)\xrightarrow{e}(A,e)$ and $(B,f)\xrightarrow{f}(B,f)$ of $\mathscr{B}$ gives

$$f\mu_{(A,e),(B,f)}(u_1,\dots,u_n)e=\mu_{(A,e),(B,f)}(fu_1e,\dots,fu_ne). \tag{12}$$

Since the left-hand sides of (11) and (12) are equal, so too are the right-hand sides; but this is the desired (10), since $fu_ie=u_i$ for $u_i\in\mathscr{B}((A,e),(B,f))$. □

Remark 5. Proposition 8 may lead the reader to wonder whether the theory-map $Y^*:[\mathscr{A}^{\mathrm{op}},\mathbf{Set}]^*\to\mathscr{A}^*$ induced by the fully faithful Yoneda embedding $Y:\mathscr{A}\to[\mathscr{A}^{\mathrm{op}},\mathbf{Set}]$ is also invertible. This is not in fact the case; for we shall see in the next section that $[\mathscr{A}^{\mathrm{op}},\mathbf{Set}]^*$, like $\mathscr{B}^*$ for any complete or cocomplete $\mathscr{B}$, is a *commutative* theory, while the general $\mathscr{A}^*$ is not commutative as we now show.

Example 6. The theory $\mathscr{A}^*$ is not commutative when $\mathscr{A}$ is the monoid $M=\{1,e\}$ with $e^2=e$.

Proof. An element of $\mathscr{A}^*(2,1)$ is a family

$$\mu_{A,\,B}:\mathscr{A}(A,\,B)\times\mathscr{A}(A,\,B)\to\mathscr{A}(A,\,B)$$

natural in A and in B; that is to say, a function $M\times M\xrightarrow{\mu}M$ satisfying $\mu(ex,ey)=e\mu(x,y)$, which is clearly equivalent to $\mu(e,e)=e$. One such μ is given by

$$\mu(1,1)=\mu(1,e)=\mu(e,1)=1,\quad \mu(e,e)=e;$$

and another — which we call v — by

$$v(1,1)=1,\quad v(1,e)=v(e,1)=v(e,e)=e.$$

If $\mathscr{A}^*$ were commutative we should have

$$\mu(v(x_1,y_1),v(x_2,y_2))=v(\mu(x_1,x_2),\mu(y_1,y_2))$$

for all x_1,y_1,x_2,y_2; but in fact we have

$$\mu(v(e,1),v(1,e))=\mu(e,e)=e,\quad v(\mu(e,1),\mu(1,e))=v(1,1)=1.\quad \square$$

Remarks 7. (a) It is easy to describe the monoid $\mathscr{A}^*(1,1)$ of unary operations in $\mathscr{A}^*$. By the Yoneda lemma every natural transformation $\gamma_{AB}:\mathscr{A}(A,\,B)\to\mathscr{A}(A,\,B)$ is of the form $\mathscr{A}(A,\alpha_B)$ for a unique natural transformation $\alpha_B : B \to B$; that is, for a unique endomorphism $\alpha : 1_{\mathscr{A}} \to 1_{\mathscr{A}}$ in $[\mathscr{A},\mathscr{A}]$. So $\mathscr{A}^*(1,1)$ is identified with the centre $\mathscr{Z}(\mathscr{A})$ of the category $\mathscr{A}$; and in particular it is a *commutative* monoid. Indeed, each unary operation $\gamma = \mathscr{A}(1,\alpha)$ of $\mathscr{A}^*$ lies in the centre $Z(\mathscr{A}^*)$ of the theory $\mathscr{A}^*$; for condition (5) applied to $\mu \in \mathscr{A}^*(n,1)$ and to $\gamma = \mathscr{A}(1,\alpha) \in \mathscr{A}^*(1,1)$ reduces to $\alpha_B\mu(f_1,\dots,f_n) = \mu(\alpha_B f_1,\dots,\alpha_B f_n)$ where $f_i \in \mathscr{A}(A,B)$; and this is indeed so by the naturality of μ.

(b) It is also easy to describe the set $\mathscr{A}^*(0,1)$ of nullary operations in $\mathscr{A}^*$. To give a natural $v_{AB} : \mathscr{A}(A,B)^0 = 1 \to \mathscr{A}(A,B)$ is to give maps $v_{AB} : A \to B$ satisfying $kv_{AB}h = v_{CD}$ for $h : C \to A$ and $k : B \to D$; which is exactly to give a system of *zero maps* in $\mathscr{A}$. Since there is at most one such system, we have $\mathscr{A}^*(0,1) = 0$ or $\mathscr{A}^*(0,1) = 1$. In the latter case, the naturality of each $\mu \in \mathscr{A}^*(n,1)$ ensures that the element v of $\mathscr{A}^*(0,1)$ lies in the centre of the theory $\mathscr{A}^*$.

4. The case where $\mathscr{A}$ admits finite powers or finite multiples

When the category $\mathscr{A}$ admits finite powers, we can argue as in Remark 7(a) even for n-ary operations in $\mathscr{A}^*$; for now $\mathscr{A}(A,B)^n \cong \mathscr{A}(A,B^n)$, and by the Yoneda lemma every natural transformation $\mathscr{A}(A,B^n) \to \mathscr{A}(A,B)$ is of the form $\mathscr{A}(A,\alpha_B)$ for a unique natural transformation $\alpha_B : B^n \to B$; that is, for a unique $\alpha : (1_{\mathscr{A}})^n \to 1_{\mathscr{A}}$ in $[\mathscr{A},\mathscr{A}]$. Note that α_B is given in terms of $\mu_{AB} : \mathscr{A}(A,B)^n \to \mathscr{A}(A,B)$ by $\alpha_B = \mu_{B^n,B}(p_1,\dots,p_n) \in \mathscr{A}(B^n,B)$, where the $p_i : B^n \to B$ are the (chosen) product-projections. If instead of finite powers $\mathscr{A}$ admits finite multiples, we can observe that $\mathscr{A}^{\mathrm{op}}$ admits finite powers, and appeal to Proposition 3; or equally argue directly that $\mathscr{A}(A,B)^n$ is now $\mathscr{A}(n\cdot A,B)$, so that to give a natural $\mu_{A,B} : \mathscr{A}(A,B)^n \to \mathscr{A}(A,B)$ is equally to give a natural $\beta_A : A \to n\cdot A$, where μ_{AB}, to within isomorphism, is $\mathscr{A}(\beta_A,B)$, and where $\beta_A = \mu_{A,n\cdot A}(q_1,\dots,q_n)$, these q_i being the coprojections $A \to n\cdot A$. In short:

Proposition 8. *When $\mathscr{A}$ admits finite powers we have an isomorphism of theories*

$$\mathscr{A}^* \cong \langle 1_{\mathscr{A}} \rangle,$$

so that

$$\mathscr{A}^*(n,1) \cong [\mathscr{A},\mathscr{A}]((1_{\mathscr{A}})^n, 1_{\mathscr{A}}) = \int_{A\in\mathscr{A}} \mathscr{A}(A^n,A). \tag{13}$$

Similarly, when $\mathscr{A}$ admits finite multiples, we have

$$\mathscr{A}^*(n,1) \cong \int_{A\in\mathscr{A}} \mathscr{A}(A,n\cdot A). \tag{14}$$

Proposition 9. *The theory $\langle 1_{\mathscr{A}} \rangle$ is commutative; so that the theory $\mathscr{A}^*$ is commutative if $\mathscr{A}$ admits finite powers or finite multiples, and in particular whenever $\mathscr{A}$ is complete or cocomplete.*

Proof. The A-component of (5) is the exterior of

$$\begin{tikzcd}
(A^n)^m \arrow[r, "(\omega_A)^m"] \arrow[d, "c"', "\cong"] & A^m \arrow[dr, "\theta_A"] & \\
 & & A . \\
(A^m)^n \arrow[uur, "\omega_{A^m}"] \arrow[r, "(\theta_A)^n"'] & A^n \arrow[ur, "\omega_A"'] &
\end{tikzcd}$$

Since the quadrangle commutes by the naturality of ω, it remains to show that the triangle commutes. For a general $B \in \mathscr{A}$, write $q_j^B : B^m \to B$ for the jth projection. Since c^{-1} is defined by $q_j^{A^n} c^{-1} = (q_j^A)^n$, we have

$$(q_j^A)^n c = q_j^{A^n}. \tag{15}$$

Now

$$\begin{aligned}
q_j^A \omega_{A^m} c &= \omega_A (q_j^a)^n c \quad \text{by the naturality of } \omega \\
&= \omega_A q_j^{A^n} \quad \text{by (15)} \\
&= q_j^A (\omega_A)^m \quad \text{by the naturality of } q_j,
\end{aligned}$$

giving the desired commutativity of the triangle above. □

Remark 10. When $\mathscr{A}$ has finite powers or finite multiples, it has by Propositions 1, 8 and 9 a canonical enrichment over the symmetric monoidal closed category $\mathscr{A}^*$-Alg.

When $\mathscr{A}$ has finite powers it is easy to calculate $(\mathscr{A}^{\mathbf{2}})^*$; here $\mathbf{2}$ is the category $\{0 \to 1\}$, and the functor-category $\mathscr{A}^{\mathbf{2}}$ is the arrow-category whose objects are arrows $f : A \to B$ in $\mathscr{A}$ and whose morphisms $f \to g$ are commutative squares

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "v"'] & D .
\end{tikzcd} \tag{16}$$

An element of $(\mathscr{A}^{\mathbf{2}})^*(n, 1)$ assigns to each $f : A \to B$ a commutative square

$$\begin{tikzcd}
A^n \arrow[r, "\alpha_f"] \arrow[d, "f^n"'] & A \arrow[d, "f"] \\
B^n \arrow[r, "\beta_f"'] & B
\end{tikzcd} \tag{17}$$

which is natural with respect to morphisms (16) of f. When $u = f = 1_C$ in (16), the naturality imposes $\alpha_f = \alpha_{1_C} = \beta_{1_C}$; and when $v = g = 1_B$ in (16), it imposes $\beta_f = \beta_{1_B} = \alpha_{1_B}$.

It follows that in (17) we have $\alpha_f = \gamma_A$ and $\beta_f = \gamma_B$ for some natural $\gamma_A : A^n \to A$; so that in fact:

$$(\mathscr{A}^{\mathbf{2}})^* \cong \mathscr{A}^* \quad \textit{for } \mathscr{A} \textit{ with finite powers.} \tag{18}$$

Consider now $(\mathscr{A}\times\mathscr{B})^*$ where $\mathscr{A}$ and $\mathscr{B}$ admit finite powers. An element of $(\mathscr{A}\times\mathscr{B})^*(n,1)$ is a pair $(\alpha_{AB} : A^n \to A, \beta_{AB} : B^n \to B)$ satisfying the naturality conditions

$$\begin{tikzcd} A^n \arrow[r, "\alpha_{AB}"] \arrow[d, "f^n"'] & A \arrow[d, "f"] & B^n \arrow[r, "\beta_{AB}"] \arrow[d, "g^n"'] & B \arrow[d, "g"] \\ C^n \arrow[r, "\alpha_{CD}"'] & C, & D^n \arrow[r, "\beta_{CD}"'] & D. \end{tikzcd} \tag{19}$$

Since $\mathscr{A}$ and $\mathscr{B}$ have finite powers — including a terminal object given by the nullary powers — each is connected. Then taking $f = 1_A$ in (19) shows α_{AB} to be independent of B, and similarly β_{AB} is independent of A; it follows that $\alpha_{AB} = \alpha_A$ and $\beta_{AB} = \beta_B$ for some $\alpha \in \mathscr{A}^*(n,1)$ and some $\beta \in \mathscr{B}^*(n,1)$. Thus we have

$$(\mathscr{A}\times\mathscr{B})^*(n,1) = \mathscr{A}^*(n,1)\times\mathscr{B}^*(n,1). \tag{20}$$

In fact, the category **Th** admits products, the product theory $\mathscr{T}\times\mathscr{S}$ having $(\mathscr{T}\times\mathscr{S})\ (n,m) = \mathscr{T}(n,m)\times\mathscr{S}(n,m)$; so that we can rewrite (20) as

$$(\mathscr{A}\times\mathscr{B})^* = \mathscr{A}^*\times\mathscr{B}^* \quad \textit{for } \mathscr{A} \textit{ and } \mathscr{B} \textit{ with finite powers.} \tag{21}$$

Since $\mathscr{A}\times\mathscr{B}$ admits a canonical enrichment over $(\mathscr{A}^*\times\mathscr{B}^*)$-Alg, we have the problem of identifying $(\mathscr{T}\times\mathscr{S})$-Alg, at least when the theories $\mathscr{T}$ and $\mathscr{S}$ are commutative. Since we have not seen this treated in the literature, we give an identification below; the reader will have no trouble extending the result to the case of an n-ary product $\mathscr{T}_1\times\mathscr{T}_2\times\cdots\times\mathscr{T}_n$. To avoid complicating the general result let us leave aside the easy cases where one or both of $\mathscr{T}$ and $\mathscr{S}$ is a *degenerate* theory: recall that $\mathscr{T}$ is degenerate when $j_{\mathscr{T}} : \mathbf{S}^{\mathrm{op}} \to \mathscr{T}$ is not faithful, and that there are two degenerate theories, one with $\mathscr{T}$-Alg $\cong \mathbf{1}$ and the other with $\mathscr{T}$-Alg $\cong \mathbf{2}$. Recall further that a commutative theory $\mathscr{T}$ has at most one nullary operation; when it has none, the initial $\mathscr{T}$-algebra, which is the free algebra $\mathscr{T}(0,1)$ on the empty set, is empty; and when $\mathscr{T}$ has one nullary operation, the initial $\mathscr{T}$-algebra $\mathscr{T}(0,1)$ is a singleton, coinciding with the terminal algebra, so that $\mathscr{T}$-Alg is a *pointed* category. In any case, we write $(\mathscr{T}\text{-Alg})'$ for the full subcategory of $\mathscr{T}$-Alg given by the non-empty algebras.

Proposition 11. *Let $\mathscr{T}$ and $\mathscr{S}$ be non-degenerate commutative theories, so that $\mathscr{T}\times\mathscr{S}$ is another such. If both $\mathscr{T}$ and $\mathscr{S}$ have nullary operations, we have*

$$(\mathscr{T}\times\mathscr{S})\text{-Alg} \cong \mathscr{T}\text{-Alg}\times\mathscr{S}\text{-Alg}.$$

Otherwise $(\mathscr{T}\times\mathscr{S})$-Alg *is obtained from* $((\mathscr{T}\times\mathscr{S})\text{-Alg})'$ *by freely adding an initial object, and* $((\mathscr{T}\times\mathscr{S})\text{-Alg})'$ *is itself isomorphic to* $(\mathscr{T}\text{-Alg})'\times(\mathscr{S}\text{-Alg})'$.

Proof. It is convenient to argue in terms of the finitary monads $T = U^{\mathscr{T}}F^{\mathscr{T}}$ and $S = U^{\mathscr{S}}F^{\mathscr{S}}$ associated to the theories $\mathscr{T}$ and $\mathscr{S}$. We have adjunctions

$$\mathscr{T}\text{-Alg}\times\mathscr{S}\text{-Alg} \underset{F^{\mathscr{T}}\times F^{\mathscr{S}}}{\overset{U^{\mathscr{T}}\times U^{\mathscr{S}}}{\rightleftarrows}} \mathbf{Set}\times\mathbf{Set} \underset{\Delta}{\overset{\Pi}{\rightleftarrows}} \mathbf{Set}, \tag{22}$$

where $\Pi(X,Y) = X\times Y$ and $\Delta X = (X,X)$. Write P for the monad in **Set** arising from the composite adjunction $(F^{\mathscr{T}}\times F^{\mathscr{S}})\Delta \dashv \Pi(U^{\mathscr{T}}\times U^{\mathscr{S}})$; we have

$$PX = TX\times SX, \tag{23}$$

while the unit $\eta^P : X \to PX$ and the multiplication $\mu^P : PPX \to PX$ are the composites

$$X \xrightarrow{\delta} X\times X \xrightarrow{\eta^T X\times \eta^S X} TX\times SX, \tag{24}$$

$$T(TX\times SX)\times S(TX\times SX) \xrightarrow{Tp_1\times Sp_2} TTX\times SSX \xrightarrow{\mu^T X\times\mu^S X} TX\times SX. \tag{25}$$

Since finite limits in **Set** commute with filtered colimits, $P = T\times S$ is (like T and S) a *finitary* monad. When X in (23) is the finite set n, we have $Pn = Tn\times Sn = \mathscr{T}(n,1)\times\mathscr{S}(n,1)$; accordingly P is *precisely* the finitary monad $U^{\mathscr{P}}F^{\mathscr{P}}$ associated to the theory $\mathscr{P} = \mathscr{T}\times\mathscr{S}$, and our problem is to identify P-Alg. Write $\mathscr{A}$ for the category asserted in the proposition to be $(\mathscr{T}\times\mathscr{S})$-Alg: the category $\mathscr{T}$-Alg$\times\mathscr{S}$-Alg in the first case, or the category obtained by adding an initial object 0 to $(\mathscr{T}\text{-Alg})'\times(\mathscr{S}\text{-Alg})'$ in the second case; and write $\mathscr{B}$ for the full subcategory of **Set**$\times$**Set** consisting of those (X,Y) where X and Y are both empty or both non-empty. There is an evident forgetful functor $V : \mathscr{A} \to \mathscr{B}$, with a left adjoint $G : \mathscr{B} \to \mathscr{A}$. Since $U^{\mathscr{T}}\times U^{\mathscr{S}} : \mathscr{T}\text{-Alg}\times\mathscr{S}\text{-Alg} \to$ **Set**$\times$**Set** is monadic, it follows easily that $V : \mathscr{A} \to \mathscr{B}$ is monadic; we leave the straightforward details to the reader. Next, the functor $W : \mathscr{B} \to$ **Set** sending (X,Y) to $X\times Y$ is not only monadic, but was shown by Barr in [1] to have a stronger property: W not only creates the coequalizers of W-split pairs, but creates them as *split* coequalizers in $\mathscr{B}$ — from which it follows that WV is monadic whenever V is monadic. In particular our WV above is monadic; and the corresponding monad is at once seen to be P, so that P-Alg $\cong \mathscr{A}$ as asserted. □

5. Conditions ensuring smallness of $\mathscr{A}^*$

As we said in the Introduction, $\mathscr{A}^*$ lacks various important properties of a Lawvere theory if the sets $\mathscr{A}^*(n,1)$ fail to be small; accordingly we now present conditions ensuring their smallness.

When $\mathscr{A}$ and $\mathscr{B}$ admit finite powers and $F : \mathscr{A} \to \mathscr{B}$ preserves these, so that each canonical comparison map, $\phi_n : F(A^n) \to (FA)^n$ is invertible, then for each $A \in \mathscr{A}$ the functions

$$\mathscr{A}(A^n, A^m) \xrightarrow{F_{A^n,A^m}} \mathscr{B}(F(A^n), F(A^m)) \xrightarrow{\mathscr{B}(\phi_n^{-1},\phi_m)} \mathscr{B}((FA)^n, (FA)^m) \tag{26}$$

clearly constitute a map $\langle A\rangle \to \langle FA\rangle$ of theories. It is more convenient below to consider the dual situation, where $\mathscr{A}$ and $\mathscr{B}$ admit finite multiples and $F:\mathscr{A}\to\mathscr{B}$ preserves these. Recall from the Introduction that we write $\{A\}$ for the theory that would be $\langle A\rangle$ for $A\in\mathscr{A}^{\mathrm{op}}$; that is, $\{A\}(n,m)=\mathscr{A}(m\cdot A, n\cdot A)\cong\mathscr{A}(A,n\cdot A)^m$. Now F induces a map $\{A\}\to\{FA\}$ of theories.

Suppose now that $\mathscr{A}$ admits finite multiples. Then so too does any functor-category $[\mathscr{C},\mathscr{A}]$, the finite multiples therein being formed pointwise. In particular $[\mathscr{A},\mathscr{A}]$ admits finite multiples, and Eq. (14) asserts that $\mathscr{A}^*\cong\{1_{\mathscr{A}}\}$. Since, for any functor $Z:\mathscr{C}\to\mathscr{A}$, the induced $[Z,1]:[\mathscr{A},\mathscr{A}]\to[\mathscr{C},\mathscr{A}]$ preserves finite multiples, it gives us a theory-map $\zeta:\{1_{\mathscr{A}}\}\to\{Z\}$, with components

$$\zeta_n:\quad \mathscr{A}^*(n,1)=\int_{A\in\mathscr{A}}\mathscr{A}(A,n\cdot A)\to\int_{C\in\mathscr{C}}\mathscr{A}(ZC,n\cdot ZC), \tag{27}$$

where ζ_n sends the natural transformation $\beta=(\beta_A:A\to n\cdot A)$ to its restriction $\beta Z=(\beta_{ZC}:ZC\to n{\cdot}ZC)$. Write $\tilde{Z}:\mathscr{A}\to[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$ for the functor given by $(\tilde{Z}A)C=\mathscr{A}(ZC,A)$, which exists whenever $\mathscr{A}$ is locally small; and recall that $\tilde{Z}$ is faithful if and only if the objects ZC for $C\in\mathscr{C}$ constitute a *generating set* for $\mathscr{A}$, while $\tilde{Z}$ is fully faithful if and only if Z is *dense* (which some call *left adequate*). It is immediate that ζ_n is equal to the composite

$$\begin{aligned}&\int_A\mathscr{A}(A,n\cdot A)\xrightarrow{\int_A\tilde{Z}_{A,n\cdot A}}\int_A[\mathscr{C}^{\mathrm{op}},\mathbf{Set}](\tilde{Z}A,\tilde{Z}(n\cdot A))\\ &\quad\cong\int_{A,C}\mathbf{Set}(\mathscr{A}(ZC,A),\mathscr{A}(ZC,n\cdot A))\cong\int_{C\in\mathscr{C}}\mathscr{A}(ZC,n\cdot ZC),\end{aligned}\tag{28}$$

wherein the last step is the Yoneda isomorphism. Moreover the arrow $\int_A\tilde{Z}_{A,n\cdot A}$ here is monomorphic when $\tilde{Z}$ is faithful, and invertible when $\tilde{Z}$ is fully faithful; while $\{Z\}(n,1)=\int_{C\in\mathscr{C}}\mathscr{A}(ZC,n\cdot ZC)$ is a small set when $\mathscr{A}$ is locally small and $\mathscr{C}$ is small. Summing up:

Proposition 12. *Consider a functor $Z:\mathscr{C}\to\mathscr{A}$ with the locally small $\mathscr{A}$ admitting finite multiples and with $\mathscr{C}$ small, so that $\{Z\}$ is a small theory with*

$$\{Z\}(n,1)=\int_{C\in\mathscr{C}}\mathscr{A}(ZC,n\cdot ZC).$$

Then the theory-map $\zeta:\mathscr{A}^*=\{1_{\mathscr{A}}\}\to\{Z\}$ is an isomorphism if Z is dense, and has monomorphic components ζ_n if the ZC form a generating set for $\mathscr{A}$.

Since we can take for Z the inclusion of any small generating set, this proposition along with Proposition 3 gives

Corollary 13. *$\mathscr{A}^*$ is a small theory if the locally small $\mathscr{A}$ admits finite multiples and has a small generating set, or admits finite powers and has a small cogenerating set.*

It is easy to calculate the right-hand side of (28), and hence to find $\mathscr{A}^*$, when $\mathscr{A}$ is the presheaf category $[\mathscr{C}^{\mathrm{op}},\mathbf{Set}]$ for some small $\mathscr{C}$ and $Z:\mathscr{C}\to\mathscr{A}$ is the Yoneda embedding. For we have

$$\begin{aligned}\int_{C\in\mathscr{C}} \mathscr{A}(ZC, n\cdot ZC) &\cong \int_C (n\cdot ZC)C \quad \text{by the Yoneda isomorphism}\\ &= \int_C (n\cdot \mathscr{C}(-,C))C\\ &= \int_C n\cdot\mathscr{C}(C,C)\\ &= \int_C n\times\mathscr{C}(C,C), \end{aligned} \tag{29}$$

this last since $n\cdot X = n\times X$ for a set X. An element of this last end is a "wedge" of vertex 1; that is, a family $(\alpha_C : 1 \to n\times\mathscr{C}(C,C))_{C\in\mathscr{C}}$ satisfying for $f:C\to D$ the naturality condition

$$\begin{tikzcd} & n\times\mathscr{C}(C,C) \arrow[dr, "n\times\mathscr{C}(C,f)"] & \\ 1 \arrow[ur, "\alpha_C"] \arrow[dr, "\alpha_D"'] & & n\times\mathscr{C}(C,D)\,. \\ & n\times\mathscr{C}(D,D) \arrow[ur, "n\times\mathscr{C}(f,D)"'] & \end{tikzcd} \tag{30}$$

This α_C has the form (β_C,γ_C) where $\beta_C : 1\to n$ and $\gamma_C : 1\to\mathscr{C}(C,C)$; and (30) becomes the two conditions

$$\beta_C = \beta_D, \tag{31}$$

$$\begin{tikzcd} & \mathscr{C}(C,C) \arrow[dr, "\mathscr{C}(C,f)"] & \\ 1 \arrow[ur, "\gamma_C"] \arrow[dr, "\gamma_D"'] & & \mathscr{C}(C,D)\,. \\ & \mathscr{C}(D,D) \arrow[ur, "\mathscr{C}(f,D)"'] & \end{tikzcd} \tag{32}$$

Here (31) holds whenever there is some map $f:C\to D$, and it asserts that β_C depends only on the *connected component* of C in $\mathscr{C}$; while (32) asserts that the γ_C: $C\to C$ are natural in C, so that the γ_C constitute an element γ of the centre $\mathscr{Z}(\mathscr{C})$ of the category $\mathscr{C}$. Thus we have shown that:

Proposition 14. $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]^* \cong n^{\pi(\mathscr{C})}\times\mathscr{Z}(\mathscr{C})$, *where* $\pi(\mathscr{C})$ *is the set of connected components of the category* $\mathscr{C}$.

Example 15. (a) Taking $\mathscr{C}=1$ here gives $\mathbf{Set}^*(n,1)=n$, so that $\mathbf{Set}^*=\mathbf{S}^{\mathrm{op}}$.

(b) When $\mathscr{C}$ is the discrete two-object category $2=\{0,1\}$, the proposition gives $(\mathbf{Set}\times\mathbf{Set})^*(n,1)=n^2$, which is consistent with (20).

(c) When $\mathscr{C}$ is the arrow-category $\mathbf{2}=\{0\to 1\}$, we have $\pi(\mathscr{C})=\mathscr{Z}(\mathscr{C})=1$, and the proposition gives $(\mathbf{Set}^{\mathbf{2}})^*(n,1)=n$, consistently with (18).

(d) We have $[(\mathscr{C}+\mathscr{D})^{\mathrm{op}},\mathbf{Set}] \cong [\mathscr{C}^{\mathrm{op}},\mathbf{Set}]\times[\mathscr{D}^{\mathrm{op}},\mathbf{Set}]$; from which (21) gives $[(\mathscr{C}+\mathscr{D})^{\mathrm{op}},\mathbf{Set}]^* \cong [\mathscr{C}^{\mathrm{op}},\mathbf{Set}]^*\times[\mathscr{D}^{\mathrm{op}},\mathbf{Set}]^*$; this is consistent with Proposition 14 since $\pi(\mathscr{C}+\mathscr{D})\cong\pi(\mathscr{C})+\pi(\mathscr{D})$ and $\mathscr{Z}(\mathscr{C}+\mathscr{D})\cong\mathscr{Z}(\mathscr{C})\times\mathscr{Z}(\mathscr{D})$.

Consider now the special case of Proposition 12 where $\mathscr{C}$ is the unit category 1, so that $Z:1\to\mathscr{A}$ merely names an object I of $\mathscr{A}$, and $\{Z\}(n,1)=\mathscr{A}(I,n\cdot I)$. If we write $V:\mathscr{A}\to\mathbf{Set}$ for the representable $\mathscr{A}(I,-)$, the Yoneda isomorphism gives

$$
\begin{aligned}
\mathscr{A}(I,n\cdot I) &\cong [\mathscr{A},\mathbf{Set}](\mathscr{A}(n\cdot I,-),\mathscr{A}(I,-))\\
&\cong [\mathscr{A},\mathbf{Set}](V^n,V)=\langle V\rangle(n,1).
\end{aligned}
\tag{33}
$$

Thus:

Proposition 16. *Let the locally small $\mathscr{A}$ admit finite multiples, let $Z:1\to\mathscr{A}$ be the name of an object I of $\mathscr{A}$, and write $V:\mathscr{A}\to\mathbf{Set}$ for the representable functor $\mathscr{A}(I,-)$. Then $\{Z\}\cong\langle V\rangle$, so that the theory-map of Proposition* 12 *takes the form $\zeta:\mathscr{A}^*\to\langle V\rangle$; and the components $\zeta_n:\mathscr{A}^*(n,1)\to\langle V\rangle(n,1)$ of this are monomorphic when I is a generator of $\mathscr{A}$ — that is, when V is faithful. Explicitly, ζ_n sends $\beta\in\int_A\mathscr{A}(A,n\cdot A)$ to $\gamma:V^n\to V$ with components $\gamma_A:\mathscr{A}(I,A)^n\to\mathscr{A}(I,A)$ where $\gamma_A(f_1,\dots,f_n)$ is the composite*

$$
I\xrightarrow{\beta_I}I+I+\cdots+I\xrightarrow{(f_1,\dots,f_n)}A.
\tag{34}
$$

Remark 17. Since the functor **Str**, unlike **Sem**, is not fully faithful, it is not automatic that the theory-map $\zeta:\mathscr{A}^*=\langle H\rangle\to\langle V\rangle$ of Proposition 16 arises by applying **Str** to a map $V\to H$ in **CAT**/**Set**. However this is the case, a suitable map being

$$
\begin{tikzcd}
\mathscr{A}=1\times\mathscr{A} \arrow[rr, "I\times\mathscr{A}"] \arrow[dr, "V"'] & & \mathscr{A}^{\mathrm{op}}\times\mathscr{A} \arrow[dl, "H"] \\
& \mathbf{Set}. &
\end{tikzcd}
\tag{35}
$$

6. Fully faithful dense functors preserving finite multiples

When $\mathscr{C}$ as well as $\mathscr{A}$ in Proposition 12 admits finite multiples, the canonical comparison $n\cdot ZC\to Z(n\cdot C)$ induces a functor $\xi_n:\{Z\}(n,1)=\int_C\mathscr{A}(ZC,n\cdot ZC)\to\int_C\mathscr{A}(ZC,Z(n\cdot C))$. When, moreover, Z is fully faithful, we have an isomorphism $\eta_n:\mathscr{C}^*(n,1)=\int_C\mathscr{C}(C,n\cdot C)\cong\int_C\mathscr{A}(ZC,Z(n\cdot C))$, and also the restriction map $Z^*:\mathscr{A}^*\to\mathscr{C}^*$

of Section 3 above; and then it is immediate that we have commutativity in

$$\begin{tikzcd} \mathscr{A}^*(n,1) \arrow[r, "\zeta_n"] \arrow[d, "Z^*_{n,1}"'] & \int_C \mathscr{A}(ZC, n\cdot ZC) \arrow[d, "\xi_n"] \\ \mathscr{C}^*(n,1) \arrow[r, "\cong", "\eta_n"'] & \int_C \mathscr{A}(ZC, Z(n\cdot C)). \end{tikzcd} \tag{36}$$

If now, in addition, Z preserves finite multiples, so that ξ_n is invertible, and Z is dense, so that ζ_n is invertible by Proposition 26, we have $Z^*_{n,1}$ invertible. That is:

Proposition 18. *Let the locally small $\mathscr{A}$ admit finite multiples, and let $\mathscr{C}$ be a dense full subcategory of $\mathscr{A}$ which is closed in $\mathscr{A}$ under finite multiples. Then the map $Z^*: \mathscr{A}^* \to \mathscr{C}^*$ induced by the inclusion $Z: \mathscr{C} \to \mathscr{A}$ is an isomorphism of theories. In particular $\mathscr{A}^*$ is a small theory when $\mathscr{C}$ is a small category.*

Remark 19. If the locally small $\mathscr{A}$ admits finite multiples and has a small dense subcategory $\mathscr{D}$, Proposition 18 gives $\mathscr{A}^* \cong \mathscr{C}^*$, where $\mathscr{C}$ is the closure of $\mathscr{D}$ under finite multiples; which is again small. Note that when $\mathscr{A}$ is locally finitely presentable, we have $\mathscr{A}^* \cong (\mathscr{A}_f)^*$ where $\mathscr{A}_f$ is the small dense subcategory of $\mathscr{A}$ given by the finitely presentable objects.

For any monad $T = (T, \eta, \mu)$ on **Set**, it is well known that the full subcategory $\mathbf{Set}_T$ of T-Alg given by the free algebras $FX = (TX, \mu X)$ is dense; and of course the free algebras satisfy $F(n \cdot X) \cong n \cdot FX$. Moreover, when T is the finitary monad corresponding to the theory $\mathscr{T}$, the full subcategory given by the free algebras Fn with n finite is already dense in T-Alg $= \mathscr{T}$-Alg, and this subcategory is clearly isomorphic to $\mathscr{T}^{\mathrm{op}}$. Putting this together with Proposition 3, we conclude that:

Proposition 20. *For any monad T on* **Set** *we have* $(T\text{-Alg})^* \cong (\mathbf{Set}_T)^*$; *and for any theory $\mathscr{T}$ we have* $(\mathscr{T}\text{-Alg})^* \cong \mathscr{T}^*$.

It is easy to describe the $\mathscr{T}^*$ of this proposition. An element α of $\mathscr{T}^*(n,1) = \int_{m \in \mathscr{T}} \mathscr{T}(m^n, m)$ is a family $(\alpha_m : m^n \to m)$ in $\mathscr{T}$ which is natural in $m \in \mathscr{T}$. A simple Yoneda-type argument shows that to give such a family is precisely to give an $\omega \in \mathscr{T}(n,1)$ which commutes, in the sense of (5), with every operation θ of $\mathscr{T}$; that is, to give an element $\omega \in \mathscr{T}(n,1)$ in the centre of $\mathscr{T}$. We conclude that:

Proposition 21. *For any theory $\mathscr{T}$, the theory* $(\mathscr{T}\text{-Alg})^* \cong \mathscr{T}^*$ *is the centre $Z(\mathscr{T})$ of the theory $\mathscr{T}$.*

Remark 22. If we take for $V: \mathscr{A} \to \mathbf{Set}$ the faithful representable functor $U^{\mathscr{T}}$: $\mathscr{T}$-Alg $\to$ **Set** for a theory $\mathscr{T}$, we have by Proposition 16 a monomorphism of theories $\zeta: (\mathscr{T}\text{-Alg})^* \to \langle U^{\mathscr{T}} \rangle$. But $\langle U^{\mathscr{T}} \rangle = \mathscr{T}$, as we saw in the Introduction, and we have just seen that $(\mathscr{T}\text{-Alg})^* \cong \mathscr{T}^* = Z(\mathscr{T})$; the reader will easily verify that ζ here is just the inclusion $Z(\mathscr{T}) \to \mathscr{T}$.

7. The case of a closed category

Let $\mathscr{V}$ be the underlying ordinary category of a symmetric monoidal closed category **V**, whose tensor product, identity object, and internal hom are denoted by $\otimes, I$, and $[\,,\,]$; and write $V : \mathscr{V} \to \mathbf{Set}$ for the representable functor $\mathscr{V}(I,-)\colon \mathscr{V} \to \mathbf{Set}$. We suppose that $\mathscr{V}$ is locally small and admits finite multiples.

If **A** is a **V**-category whose underlying ordinary category is $\mathscr{A}$, we have a commutative diagram

$$\begin{tikzcd} \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \arrow[rr, "\mathrm{A}(-,-)"] \arrow[dr, "H"'] & & \mathscr{V} \arrow[dl, "V"] \\ & \mathbf{Set} & \end{tikzcd} \tag{37}$$

as in [4, (1.33)]; and applying to this the functor **Str** gives a map $\rho : \langle V \rangle \to \langle H \rangle = \mathscr{A}^*$ of theories. Now consider in particular the case where **A** is **V** itself, so that (37) becomes

$$\begin{tikzcd} \mathscr{V}^{\mathrm{op}} \times \mathscr{V} \arrow[rr, "{[\,,\,]}"] \arrow[dr, "H"'] & & \mathscr{V} \arrow[dl, "V"] \\ & \mathbf{Set}\,; & \end{tikzcd} \tag{38}$$

here, besides the theory-map $\rho : \langle V \rangle \to \langle H \rangle = \mathscr{V}^*$ above, we have the theory-map $\zeta : \mathscr{V}^* \to \langle V \rangle$ of Proposition 16. Let us calculate the composite $\zeta\rho$. Consider an element γ of $\langle V \rangle(n,1)$, given by a natural transformation from $V^n \cong \mathscr{V}(n \cdot I, -)$ to $V = \mathscr{V}(I,-)$, which we may identify by Yoneda with the corresponding map $\gamma : I \to n \cdot I$ in $\mathscr{V}$. The element $\rho(\gamma)$ of $\mathscr{V}^*(n,1) = \langle H \rangle(n,1)$, seen as a natural transformation $\mathscr{A}(A,B)^n \to \mathscr{A}(A,B)$, has the components

$$\mathscr{V}(A,B)^n \cong \mathscr{V}(n \cdot I, [A,B]) \xrightarrow{\mathscr{V}(\gamma,1)} \mathscr{V}(I,[A,B]) \cong \mathscr{V}(A,B).$$

It follows from Section 4 above that $\rho(\gamma)$, when seen instead as an element β of $\int_A \mathscr{V}(A, n \cdot A)$, is that β whose components make commutative

$$\begin{tikzcd} \mathscr{V}(n \cdot I, [A,B]) \arrow[r, "\mathscr{V}(\gamma,1)"] \arrow[d, "\cong"'] & \mathscr{V}(I,[A,B]) \arrow[dd, "\cong"] \\ \mathscr{V}(A,B)^n \arrow[d, "\cong"'] & \\ \mathscr{V}(n \cdot A, B) \arrow[r, "\mathscr{V}(\beta_A, B)"'] & \mathscr{V}(A,B). \end{tikzcd} \tag{39}$$

Finally, by Proposition 16, $\zeta(\rho(\gamma))$ is $\beta_I : I \to n \cdot I$, and this is γ since we have commutativity in

$$\begin{tikzcd}
\mathscr{V}(n\cdot I, [I,B]) \arrow[rr, "\mathscr{V}(\gamma,1)"] & & \mathscr{V}(I,[I,B]) \\
\mathscr{V}(I,B)^n \arrow[u, "\cong"] & & \\
\mathscr{V}(n\cdot I, B) \arrow[u, "\cong"] \arrow[rr, "\mathscr{V}(\gamma,1)"'] & & \mathscr{V}(I,B); \arrow[uu, "\cong"']
\end{tikzcd} \tag{40}$$

to verify this it suffices, by the representability of $\mathscr{V}(n \cdot I, B)$, to put $B = n \cdot I$ and to apply both legs to the identity of $n \cdot I$: the results are easily seen to be equal. It follows that:

Proposition 23. *Let $\mathscr{V}$ be the ordinary category underlying a locally-small symmetric monoidal closed category with identity-object I, and write V for $\mathscr{V}(I,-) : \mathscr{V} \to$ **Set**. Then the theory-map $\rho : \langle V \rangle \to \mathscr{V}^*$ above and the theory-map $\zeta : \mathscr{V}^* \to \langle V \rangle$ of Proposition* 16 *satisfy $\zeta\rho = 1$. Accordingly, since ζ is monomorphic by Proposition* 12 *when V is faithful, the maps ζ and ρ are mutually inverse in that case.*

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